

Asymptotic Freedom Cosmology

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Abstract

For a general class of scalar–tensor gravity theories, we discuss how to recover asymptotic freedom regimes when cosmic time $t \rightarrow \pm\infty$. Such a feature means that the effective gravitational coupling $G_{eff} \rightarrow 0$, while cosmological solutions can asymptotically assume de Sitter or power-law behaviours. In our opinion, through this mechanism, it is possible to cure some shortcomings in inflationary and in string–dilaton cosmology.

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1 Introduction

In the last decades, people are taking into serious consideration alternative theories of gravity whose effective actions are more general than that of Hilbert–Einstein. This approach is mainly due to unification schemes which must include gravity coherently with the other interactions at a quantum level [1]. From a cosmological point of view, in order to construct more realistic models with respect to the standard one, alternative theories are needed to avoid general relativity shortcomings [2], to get inflation without fine-tuning [3], to construct perturbation spectra in agreement with the observed large-scale structure [4].

Anyway, these extended gravity theories (which generally includes higher-order terms in curvature invariants or nonminimal couplings between geometry and one or more scalar fields [5]) introduce new features that general relativity, in the Einstein formulation, does not possess. Among them, asymptotic freedom seems to emerge as a sort of new characteristic connected to singularity-free cosmological models or inflationary behaviours. Its emergence could be the result of the fact that gravity is an “induced” interaction coming from an average effect of the other fundamental forces [6].

However, gravitational asymptotic freedom is just analogue, for example, with respect to that introduced in strong interactions, due to the lack of a full quantum gravity theory.

From a classical point of view, using cosmological models derived from alternative gravity theories, it is possible to recover the fact that effective gravitational coupling goes to zero in asymptotic regimes.

Taking into account non-Abelian gauge theories (*e.g.* QCD), we have that the effective strength of interactions goes to zero at short distances when the energy diverges [7],[8]. The effect was tested by deep-inelastic scattering experiments [9],[10].

In cosmology, the corresponding feature of “short distances” could be “early times” (*i.e.* $t \rightarrow -\infty$), where energy diverges and gravitational coupling has to vary with respect to the present value in order to recover something similar to the “running coupling constant” of QCD [8]. A sort of asymptotic freedom could be recovered also toward the future ($t \rightarrow +\infty$) as soon as $G_{eff} \rightarrow 0$.

Several papers have recently been devoted to such a problem. Among them, some are searching for the effect of gravitational asymptotic freedom in the large scale structure. In fact, it is possible to show that a scale-dependent gravitational coupling affects the two-point correlation function and the Jeans length for structure formation [11].

It is possible to recover asymptotic freedom also in higher-order gravity [12]. In this approach, a limiting curvature appears at early times and a dynamically consistent picture is possible at the Planck epoch [13].

Also in string-dilaton gravity, it is possible to recover an asymptotic-freedom regime depending on the number of dimensions of the theory [14].

In a previous paper [15], two of the authors faced the problem to obtain cosmological asymptotic freedom in scalar-tensor theories of gravity. It was shown that in a Friedman–Robertson–Walker (FRW) flat spacetime, we can obtain singularity-free solutions where

the effective gravitational coupling $G_{eff} \rightarrow 0$ for $t \rightarrow -\infty$. For some of the solutions, we got $G_{eff} \rightarrow G_N$ for $t \rightarrow +\infty$, where G_N is the Newton constant. Such features are recovered if cosmological models satisfy a set of general conditions fixed *a priori*. In this paper, we generalize those results asking for asymptotic freedom toward $t \rightarrow \pm\infty$ and searching for those classes of models, with asymptotic behaviours for the scale factor $a(t)$ evolving as $a(t) \simeq e^{\alpha t}$ and $a(t) \simeq t^p$.

In Sec.2, we derive the cosmological equations of motion for a general class of scalar–tensor theories. Sec.3 is devoted to recover a differential equation for the gravitational coupling by which it is possible to discuss the asymptotic freedom. In Secs.4 and 5, asymptotic freedom is recovered for exponential and power–law cosmological behaviours. Discussion and conclusions are drawn in Sec.6.

2 Scalar–tensor FRW cosmology

We start using the following action

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[F(\phi)R + \frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - V(\phi) + \mathcal{L}_m \right], \quad (1)$$

where $F(\phi)$, $V(\phi)$ are generic functions of ϕ and \mathcal{L}_m is the ordinary matter Lagrangian density. In a FRW spacetime, the equations of motions are

$$H^2 + \left(\frac{\dot{F}}{F} \right) H + \frac{k}{a^2} + \frac{\rho_\phi}{6F} + \frac{\rho_m}{6F} = 0, \quad (2)$$

$$2\dot{H} + 3H^2 + 2 \left(\frac{\dot{F}}{F} \right) H + \frac{\ddot{F}}{F} + \frac{k}{a^2} - \frac{p_\phi}{2F} - \frac{p_m}{2F} = 0, \quad (3)$$

$$\ddot{\phi} + 3H\dot{\phi} + 12F'H^2 + 6F'\dot{H} + 6F'\frac{k}{a^2} + V' = 0, \quad (4)$$

while the fluid–matter dynamics is given by the conservation equation

$$\dot{\rho}_m + 3H(p_m + \rho_m) = 0; \quad (5)$$

and, as usual, the state equation is given by

$$p_m = (\gamma - 1)\rho_m, \quad 1 \leq \gamma \leq 2. \quad (6)$$

We have that $H = \frac{\dot{a}}{a}$ is the Hubble parameter, $\dot{F} = \frac{dF}{d\phi}\dot{\phi} = F'\dot{\phi}$ is the time derivative of coupling $F(\phi)$,

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \quad (7)$$

are, respectively, the energy density and pressure of the scalar field ϕ while ρ_m and p_m are the corresponding quantities for ordinary matter; γ is a constant.

Usually, the system of differential equation (2)–(6) is solved for $a(t)$ and $\phi(t)$ [16], given $F(\phi)$, $V(\phi)$, γ , and k ; one gets then $a(t)$, $\phi(t)$ and ρ_ϕ , p_ϕ , ρ_m , p_m .

On the other side, we can assign the behaviours of $a(t)$ and $V(\phi(t))$ and look for some particular behaviour of $F(\phi(t))$ and $\phi(t)$. However, the constants γ and k have to be given.

We shall face this second approach since we want to ask for asymptotic freedom, *i.e.* the behaviour of $F(\phi) \rightarrow \infty$, for cosmological solutions of the forms $a(t) \sim e^{\alpha t}$ and $a(t) \sim t^p$. In fact, if asymptotic freedom is recovered, the gravitational coupling

$$G_{eff} = -\frac{1}{2F(\phi)}, \quad (8)$$

goes to zero. Eq.(8) is written in Planck's units. The standard Newtonian coupling $8\pi G_N = 1$ is recovered for $F(\phi) = -\frac{1}{2}$.

3 Recovering gravitational asymptotic freedom

As first issue, we have to obtain a differential equation for $F(\phi(t))$ which analysis could give rise to the classes of models showing asymptotic freedom. Let us consider the Einstein equations (2) and (3). Combining them, with a little algebra, we get

$$\ddot{F} + 5H\dot{F} + 2(3H^2 + \dot{H})F + V + \frac{4k}{a^2}F - \frac{(\gamma - 2)}{2}\rho_m = 0. \quad (9)$$

Using Eqs.(5) and (6), we have

$$\rho_m = \frac{D}{a^{3\gamma}}, \quad (10)$$

where D is an integration constant. Then Eq.(9) becomes

$$\ddot{F} + 5H\dot{F} + 2(3H^2 + \dot{H})F + V + \frac{4k}{a^2}F - \frac{(\gamma - 2)}{2}\frac{D}{a^{3\gamma}} = 0. \quad (11)$$

At this point, it is necessary to assign the form of the potential V . Following the analysis we have already done in the context of the so called *Noether cosmologies* [16],[17], a simple but useful choice could be

$$V(\phi) = V_0 F(\phi)^s, \quad V(\phi) \geq 0, \quad (12)$$

s being a free parameter. We consider the case $F(\phi) < 0$ which has physical meaning. The equation for the coupling F becomes then

$$\ddot{F} + 5H\dot{F} + 2(3H^2 + \dot{H})F + V_0 F^s + \frac{4k}{a^2}F - \frac{(\gamma - 2)}{2}\frac{D}{a^{3\gamma}} = 0. \quad (13)$$

It is worthwhile to note that such an equation can be invariant under time reversal ($t \rightarrow -t$) for suitable choices of the parameters V_0 and s in the sense that the functions $F(t)$, $a(t)$, and $F(-t)$, $a(-t)$ satisfy the same differential equation. This feature is interesting to get asymptotic freedom toward $t \rightarrow \pm\infty$.

4 The exponential behaviour

Let us consider, for the scale factor of the universe $a(t)$, an asymptotic behaviour of the form

$$a(t) = a_0 e^{\alpha t}, \quad \alpha > 0, \quad t \gg 0, \quad (14)$$

where α is a constant. Considerations similar to that below will hold for $t \ll 0$.

Being $H = \alpha$, $\dot{H} = 0$, we get

$$\ddot{F} + 5\alpha\dot{F} + 6\alpha^2 F + V_0 F^s + \frac{4k}{a_0^2} e^{-2\alpha t} F - \frac{(\gamma - 2)}{2} \frac{D}{a_0^{3\gamma}} e^{-3\gamma\alpha t} = 0. \quad (15)$$

A simple choice is assuming a FRW spacetime with $k = 0$. Furthermore, we are interested in asymptotic regime so that ordinary matter term has no influence and then it can be discarded. The asymptotic equation for the coupling becomes

$$\ddot{F} + 5\alpha\dot{F} + 6\alpha^2 F + V_0 F^s = 0, \quad (16)$$

which depends on the free parameter s . This ordinary differential equation is quite general and, in principle, holds any time we get an asymptotic de Sitter behaviour in a nonminimally coupled theory.

4.1 The case $s = 0$

If $s = 0$ and $V_0 \geq 0$, the general solution of Eq.(16) is

$$F(t) = A_1 e^{-2\alpha t} + A_2 e^{-3\alpha t} - \frac{V_0}{6\alpha^2}, \quad (17)$$

where $A_{1,2}$ are integration constants. It is interesting to see that, by using the definition (8),

$$F \longrightarrow -\frac{V_0}{6\alpha^2}, \quad G_{eff} \longrightarrow \frac{3\alpha^2}{V_0}, \quad (18)$$

and

$$\frac{\dot{F}}{HF} \longrightarrow 0, \quad \frac{\dot{G}_{eff}}{HG_{eff}} \longrightarrow 0, \quad (19)$$

for $t \rightarrow +\infty$, so that it is possible to recover standard gravity in such a regime. With respect to the considerations in [15], the situation here presented is new and, by it, the time-coupling variation \dot{F}/F is compared with cosmological evolution H .

Substituting solution (17) into the Klein-Gordon Eq.(4), (with $k = 0$), we get the behaviour of the field $\phi(t)$

$$\phi(t) = A_3 e^{-3\alpha t} + \phi_0, \quad (20)$$

so that the coupling is

$$F(\phi) = A_1 \left(\frac{\phi - \phi_0}{A_3} \right)^{2/3} + A_2 \left(\frac{\phi - \phi_0}{A_3} \right) - \frac{V_0}{6\alpha^2}. \quad (21)$$

Also here, A_3 is an integration constant. Standard gravity is recovered since, for $t \gg 0$,

$$F \rightarrow -\frac{V_0}{6\alpha^2} < 0. \quad (22)$$

4.2 The case $s = 1$

Another interesting choice is

$$s = 1, \quad F < 0, \quad V_0 < 0. \quad (23)$$

The solution of Eq.(16) is

$$F(t) = \exp\left(-\frac{5\alpha}{2}t\right) \left[A_1 \exp\left(\alpha\sqrt{\frac{1}{4} - \frac{V_0}{\alpha^2}}t\right) + A_2 \exp\left(-\alpha\sqrt{\frac{1}{4} - \frac{V_0}{\alpha^2}}t\right) \right]. \quad (24)$$

Asymptotically, for $t \rightarrow +\infty$, we have

$$F \rightarrow -\infty, \quad G_{eff} \rightarrow 0, \quad (25)$$

if

$$\frac{|V_0|}{6\alpha^2} > 1, \quad A_1 < 0, \quad (26)$$

and, always for $t \rightarrow +\infty$,

$$F \rightarrow 0, \quad G_{eff} \rightarrow \infty, \quad (27)$$

if

$$\frac{|V_0|}{6\alpha^2} < 1, \quad A_2 < 0. \quad (28)$$

In the case (25), we recover the asymptotic freedom. We have that

$$\frac{\dot{F}}{HF} \rightarrow -\left(\frac{5}{2} - \sqrt{\frac{1}{4} - \frac{V_0}{\alpha^2}}\right); \quad \frac{\dot{G}_{eff}}{HG_{eff}} \rightarrow \left(\frac{5}{2} - \sqrt{\frac{1}{4} - \frac{V_0}{\alpha^2}}\right). \quad (29)$$

If conditions (28) holds, the situation is analogue to the case $s = 0$. If

$$1 < \frac{|V_0|}{6\alpha^2} < 2, \quad A_1 < 0, \quad (30)$$

we get

$$F(\phi) = \xi_0 (\phi - \phi_0)^2, \quad (31)$$

where

$$\xi_0 = -\frac{\left(\frac{7}{2} + \sqrt{\frac{1}{4} - \frac{V_0}{\alpha^2}}\right) \left(-\frac{5}{2} + \sqrt{\frac{1}{4} - \frac{V_0}{\alpha^2}}\right)}{48 \left(2 + \frac{V_0}{6\alpha^2}\right)}. \quad (32)$$

Of course, in this case, the potential assume a form similar to the coupling (31). Similar analysis can be performed for other values of s . The main point is that, by imposing an asymptotic de Sitter behaviour for the scale factor $a(t)$, we can recover asymptotic freedom in the case of attractive gravity toward $t \rightarrow +\infty$. It straightforward to see that analogous results hold for $t \rightarrow -\infty$ and $a \simeq e^{-\alpha t}$, $\alpha > 0$, that is one can have asymptotic freedom for infinitely negative t .

5 The power-law behaviour

Let us now assume an asymptotic power-law behaviour for the scale-factor of the universe, that is

$$a(t) \sim t^p, \quad p > 0, \quad t \gg 0. \quad (33)$$

Eq.(13) becomes

$$\ddot{F} + \frac{5p}{t}\dot{F} + \frac{1}{t^2} \left[2p(3p-1) + \left(\frac{4kt_0^2}{a_0^2} \right) \left(\frac{t_0}{t} \right)^{2p-2} \right] F + V_0 F^s - \frac{(\gamma-2)}{2} \frac{D}{a_0^{3\gamma}} \left(\frac{t_0}{t} \right)^{3\gamma p} = 0, \quad (34)$$

being

$$a(t) = a_0 \left(\frac{t_0}{t} \right)^p, \quad H = \frac{p}{t}, \quad \dot{H} = -\frac{p}{t^2}. \quad (35)$$

Eq.(34) can be a Fuchs equation [18] depending on the values of k and p . As before, a simple choice is $k = 0$.

5.1 The case $s = 0$

If $s = 0$, for $t \rightarrow +\infty$, it is easy to see that

$$\frac{(\gamma-2)}{2} \frac{D}{a_0^{3\gamma}} \left(\frac{t_0}{t} \right)^{3\gamma p} \ll V_0, \quad (36)$$

and then Eq.(34) reduces to

$$\ddot{F} + \frac{5p}{t}\dot{F} + \frac{2p(3p-1)}{t^2}F + V_0 = 0. \quad (37)$$

The homogeneous associated equation (*i.e.* $V_0 = 0$) is a totally Fuchsian equation whose general solution is of the form

$$F(t) \sim t^r + B_1 t^2 + B_2 t + B_3, \quad (38)$$

where r has the values $r_1 = 1 - 3p$ or $r_2 = -2p$, so that

$$F(t) = A_1 t^{1-3p} + A_2 t^{-2p} + B_1 t^2 + B_2 t + B_3. \quad (39)$$

Substituting into Eq.(37), we get

$$B_1 = -\frac{V_0}{2p(3p-1)+2}, \quad B_2 = B_3 = 0. \quad (40)$$

The parameter p can assume any non-negative real value. The final solution is

$$F(t) = A_1 t^{1-3p} + A_2 t^{-2p} - \frac{V_0 t^2}{2p(3p-1)+2}. \quad (41)$$

Asymptotically, for $t \rightarrow +\infty$, we have

$$F \longrightarrow -\infty; \quad G_{eff} \longrightarrow 0, \quad (42)$$

which means the recovering of asymptotic freedom.

5.2 The case $s = 1$

Another choice can be $s = 1$ for $F < 0$ which implies $V_0 < 0$. Eq.(13) is

$$\ddot{F} + \frac{5p}{t}\dot{F} + \left[\frac{2p(3p-1)}{t^2} + V_0 \right] F - \frac{(\gamma-2)}{2} \frac{D}{a_0^{3\gamma}} \left(\frac{t_0}{t} \right)^{3\gamma p} = 0, \quad (43)$$

whose homogeneous associated equation is

$$\ddot{F} + \frac{5p}{t}\dot{F} + \left[\frac{2p(3p-1)}{t^2} + V_0 \right] F = 0, \quad (44)$$

which is not a Fuchs equation for $t \rightarrow +\infty$. We can search for a solution of the form

$$F(t) \sim e^{\beta t} t^m \sum_{n=0}^{\infty} \frac{f_n}{t^n}, \quad (45)$$

where β , m , n and f_n are arbitrary constants [18]. Inserting (45) into (44), we find

$$F(t) = t^{-(5p)/2} \left[A_1 e^{\sqrt{|V_0|}t} + A_2 e^{-\sqrt{|V_0|}t} \right], \quad (46)$$

and then, for $t \rightarrow +\infty$,

$$F \longrightarrow -\infty, \quad G_{eff} \longrightarrow 0, \quad (47)$$

if $A_1 < 0$ and p is real and non-negative.

The behaviour of $\phi(t)$ and the dependence of F by $\phi(t)$ can be recovered as above by taking into account the Klein–Gordon Eq.(4). Similar considerations hold for the other values of s .

6 Discussion and conclusions

Asymptotic freedom seems to be a common feature for general classes of scalar–tensor theories of gravity, *i.e.* when the gravitational coupling is assumed to depend on a scalar field.

Furthermore, we have shown that it can be recovered for $t \rightarrow \pm\infty$ so that, as supposed for example, in string–dilaton cosmology [19], it seems that a sort of asymptotic symmetry exists in several classes of cosmological models. It strictly depends on the relative signs of the parameters inside the scalar–field potentials and couplings, but many kinds of cosmological behaviours (*i.e.* $a(t) \sim e^{at}$ or $a(t) \sim t^p$) of physical interest can exhibit such a feature.

The main point is to obtain from the dynamical equations of a certain class of cosmological models, an ordinary differential equation which gives the behaviour of the gravitational coupling $F(\phi(t))$.

Furthermore, some models allow to recover the standard Einstein gravity (at least an attractive behaviour for the interaction) as soon as $G_{eff} \rightarrow \text{constant}$. In this sense,

general relativity is not ruled out by the presence of asymptotic freedom but it is a complementary feature for particular classes of models.

As discussed also in [15], this situation can be read as a classical analogue of QCD toward a full theory in which gravity is an interaction induced by the other non-gravitational fundamental forces.

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